On Chromatic Polynomials, List Color Functions, and DP Color Functions

Jeffrey A. Mudrock

College of Lake County
Grayslake, Illinois

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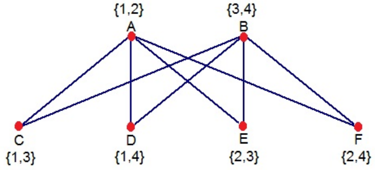
Joint work with Hemanshu Kaul

Student Co-authors

- Jack Becker, CLC (undergrad), DePaul (MS).
- Vu Bui, CLC (undergrad).
- Charlie Halberg, CLC & Utah (undergrad).
- Jade Hewitt, CLC & IIT (undergrad).
- Akash Kumar, CLC & UIUC (undergrad).
- Andrew Liu, Stevenson H. S., CLC & MIT (undergrad).
- Michael Maxfield, CLC & Wisconsin-Madison (undergrad).
- Patrick Rewers, CLC & Purdue (undergrad), GT (MS).
- Paul Shin, Stevenson H. S., CLC & Dartmouth (undergrad).
- David Spivey, CLC (undergrad).
- Seth Thomason, CLC & SIU (undergrad).
- Khue To, CLC (undergrad).
- Tim Wagstrom, CLC & UIC (undergrad), URI (PhD).

List Coloring

- For graph G a ***list assignment for*** G, L, assigns each $v \in V(G)$ a list, $L(v)$, of available colors. A ***proper L-coloring*** of G is a proper coloring, f, of G such that $f(v) \in L(v)$ for all $v \in V(G)$.



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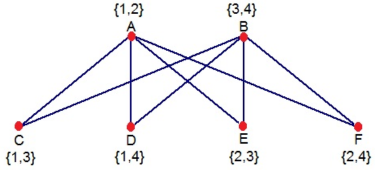

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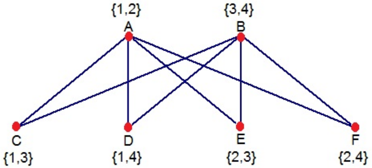

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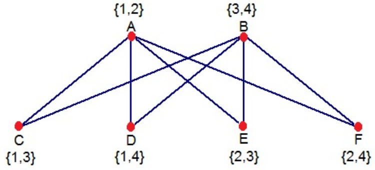

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- Clearly, $\chi(G) \leq \chi_\ell(G)$ (e.g., $2 = \chi(K_{2,4}) < \chi_\ell(K_{2,4})$).

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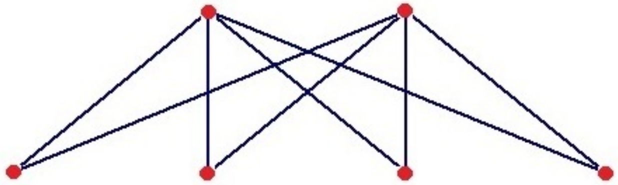
- For example, $P(K_{2,l}, m) = m(m-1)^l + m(m-1)(m-2)^l$.



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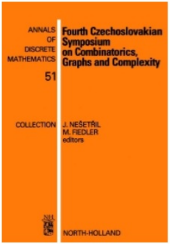

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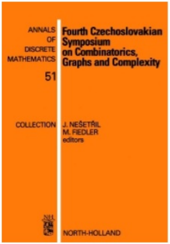

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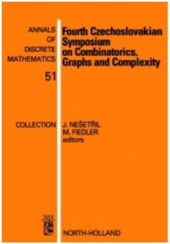

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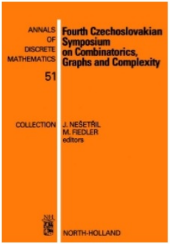

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- For example, $P_\ell(K_{2,4}, 2) = 0$, yet $P(K_{2,4}, 2) = 2$.

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Theorem (Kostochka, Sidorenko (1990); Kirov, Naimi (2016); Kaul, M. (2021))

For each $m \in \mathbb{N}$ the following statements hold.
1. $P_\ell(K_n,m) = P(K_n,m) = \prod_{i=0}^{n-1}(m-i)$.
2. $P_\ell(T,m) = P(T,m) = m(m-1)^{n-1}$, *tree T with $|V(T)| = n$.*
3. *For $n \geq 3$, $P_\ell(C_n,m) = P(C_n,m) = (m-1)^n + (-1)^n(m-1)$.*
4. *For $n \geq 3$ and $k \in \mathbb{N}$, $P_\ell(C_n \vee K_k, m) = P(C_n \vee K_k, m)$.*

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Question (Kostochka, Sidorenko (1990))

For each graph G, does there exist an $N_G \in \mathbb{N}$ such that $P_\ell(G,m) = P(G,m)$ whenever $m \geq N_G$?

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- The answer is yes! (Donner, 1992)

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For every graph G, does $\ell(G) = \tau(G)$? In other words, if $P_\ell(G, t) = P(G, t)$ for some $t \geq \chi(G)$, does it follow that $P_\ell(G, t+1) = P(G, t+1)$?

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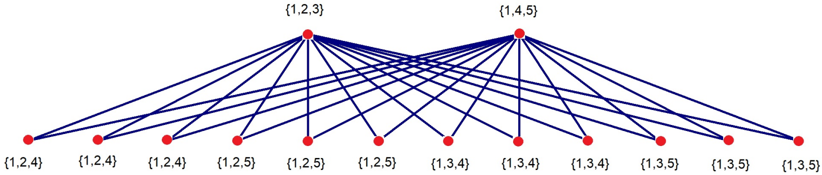
Consider the 3-assignment L for $G = K_{2,12}$.



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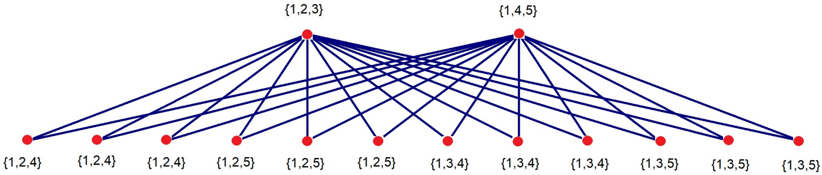

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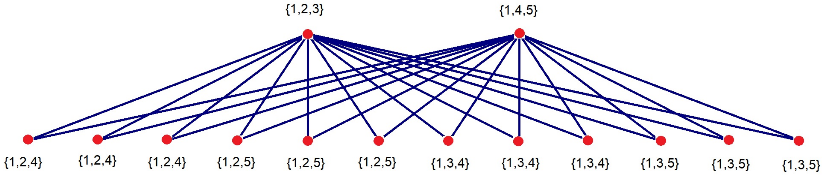

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$P(G, L) = 11264$. So, $P_\ell(G, 3) < P(G, 3)$.

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Theorem (Kaul, Kumar, M., Rewers, Shin, To (2022+))

Suppose $G = K_{2,l}$ and $l \geq 16$. Let $q = \lfloor l/4 \rfloor$. Then,

$$\tau(G) > \left\lfloor \left(\frac{q}{\ln(16/7)} \right)^{1/2} + 1 \right\rfloor.$$

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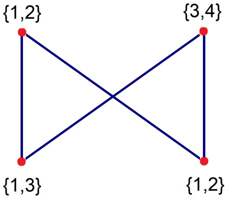
Question

Let $\delta_{max}(t) = \max\{\tau(G) - \chi_\ell(G) : |E(G)| \leq t\}$. *What is the asymptotic behavior of* $\delta_{max}(t)$*?*

DP-Coloring: A Different Perspective

- Assume the list assignment below is L. Suppose we want to know if there is a proper L-coloring of the graph below.



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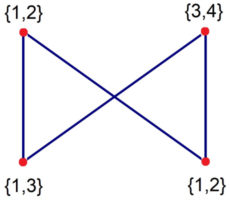
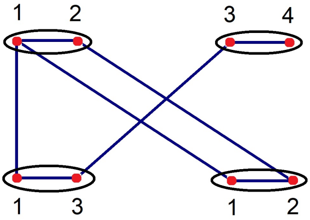


- Equivalent to finding an independent set of size 4 in:



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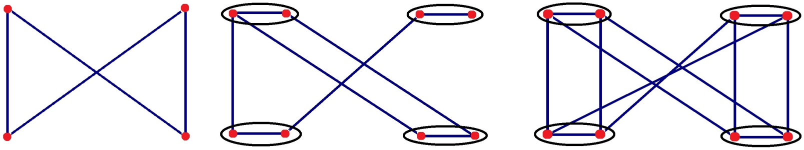

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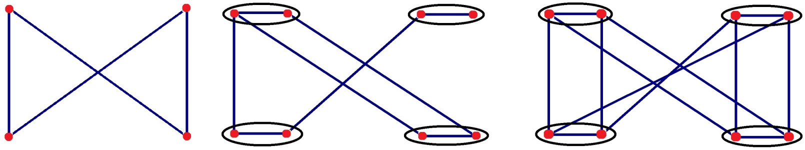

- We say $\mathcal{H}$ is *m-fold* if $|L(u)| = m$ for each $u \in V(G)$.

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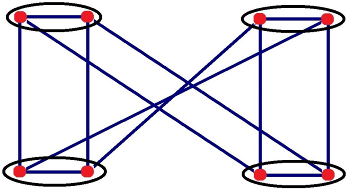

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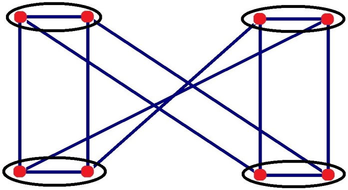

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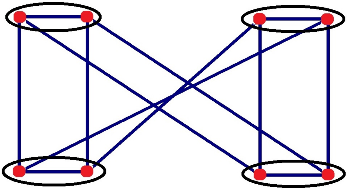

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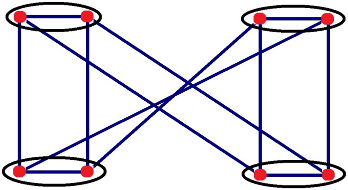

- In general, $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

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Suppose G is a unicyclic graph on n vertices. For $m \geq 2$, if G contains a cycle on $2k + 2$ vertices, then $P_{DP}(G, m) = (m-1)^n - (m-1)^{n-2k-2} < P(G, m)$.

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For graph G let $\ell_G : E(G) \to \mathbb{N} \cup \{\infty\}$ be the function that maps each cut-edge in G to ∞ and maps each non-cut-edge $e \in E(G)$ to the length of a shortest cycle in G containing e. If G contains an edge l such that $\ell_G(l)$ is even, then there exists $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.

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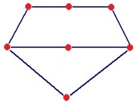

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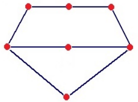

Theorem (Halberg, Kaul, Liu, M., Shin, Thomason (2020+))

Suppose that G is a graph with a feedback vertex set of size one. Then there exists $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ for all $m \geq N$.

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For every graph G, does $\ell(G) = \tau(G)$? In other words, if $P_\ell(G,t) = P(G,t)$ for some $t \geq \chi(G)$, does it follow that $P_\ell(G,t+1) = P(G,t+1)$?

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Theorem (Bui, Kaul, Maxfield, M., Shin, Thomason (2021+))

If G is $\Theta(2,3,3,3,2)$ or $\Theta(2,3,3,3,3,3,2,2)$, then $P_{DP}(G,3) = P(G,3)$ and there is an $N \in \mathbb{N}$ such that $P_{DP}(G,m) < P(G,m)$ for all $m \geq N$.

Finite DP Color Function Thresholds

Theorem (Dong, Yang (2022))

For graph G suppose ℓ_G maps each non-cut-edge $e \in E(G)$ to the length of a shortest cycle in G containing e. If G contains a spanning tree T such that for each $e \in E(G) - E(T)$,
(i) $\ell_G(e)$ is odd and
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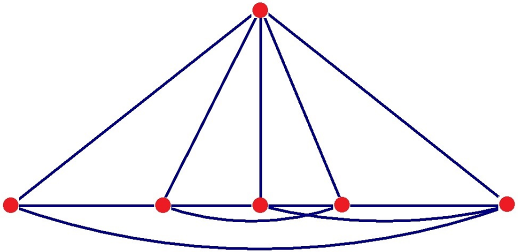

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For any $p \in \mathbb{N}$ and $n \geq 3$, $\tau_{DP}(K_p \vee C_n) = 3 + p$.

Recall $\chi(K_p \vee C_{2k+2}) = 2 + p$ and $\chi(K_p \vee C_{2k+1}) = 3 + p$.

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Theorem (Becker et. al. (2021+))

Let $M = K_1 \vee G$*, where* G *is the disjoint union of cycles* C_{k_i} *for* $i \in [n]$*, with each* $k_i \geq 3$*. Then,*

$$\tau_{DP}(M) = \begin{cases} 5 & \textit{if } \exists \textit{ distinct } i,j \in [n] \textit{ such that } k_i = k_j = 4 \\ 4 & \textit{otherwise.} \end{cases}$$

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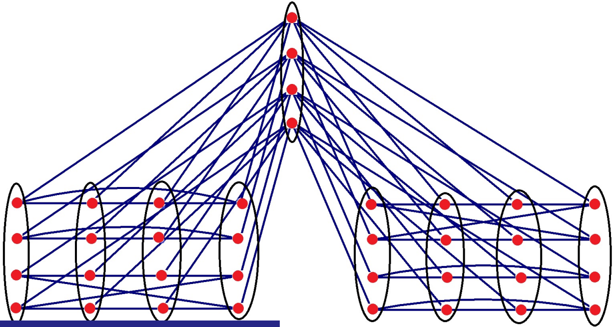

Anyone Have Questions or... Answers?

1. If $P_\ell(G,t) = P(G,t)$ for some $t \geq \chi(G)$, does it follow that $P_\ell(G,t+1) = P(G,t+1)$?

2. Let $\delta_{max}(t) = \max\{\tau(G) - \chi_\ell(G) : |E(G)| \leq t\}$. What is the asymptotic behavior of $\delta_{max}(t)$?

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