DP-Coloring the Cartesian Product of Graphs

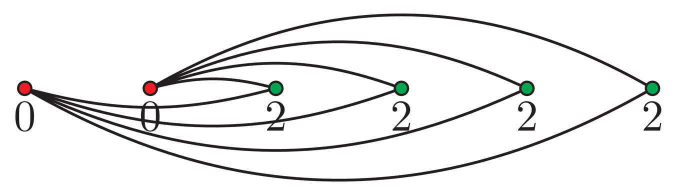
ILLINOIS INSTITUTE OF TECHNOLOGY

Hemanshu Kaul, Jeffrey Mudrock, Gunjan Sharma, and Quinn Stratton

1. Preliminaries

An important problem in graph theory is conflict-free allocation of limited resources. Suppose we want to assign time-slots (a limited resource) to committees so that committees having a common member get different time-slots (the conflict relationship). A *proper k-coloring* of a graph G is a function $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ if $uv \in E(G)$.

- The *chromatic number* $\chi(G)$ is the smallest k so that there exists a proper k-coloring of G.
- The *coloring number* $\text{col}(G)$ is the smallest k such that there exists an ordering of the vertices of G where each vertex has at most $k-1$ neighbors preceding it. This gives a classic greedy upper bound on $\chi(G)$.



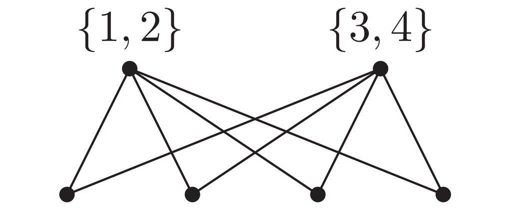
$\text{col}(K_{2,4}) = 3$
$\chi(K_{2,4}) = 2$

$K_{2,4}$

A *list assignment* for G assigns a list of available colors to each vertex in $V(G)$.

- A proper coloring f is called an *L-coloring* if $f(v) \in L(v)$ for each $v \in V(G)$. If $|L(v)| = k$ for all $v \in V(G)$ we call L a *k-assignment*.
- The *list chromatic number* $\chi_\ell(G)$ is the least k such that G admits an L-coloring for every k-assignment L.

The example below shows that $\chi_\ell(K_{2,4}) > 2$.



Note: $\chi(G) \leq \chi_\ell(G) \leq \text{col}(G) \leq \Delta(G) + 1$.

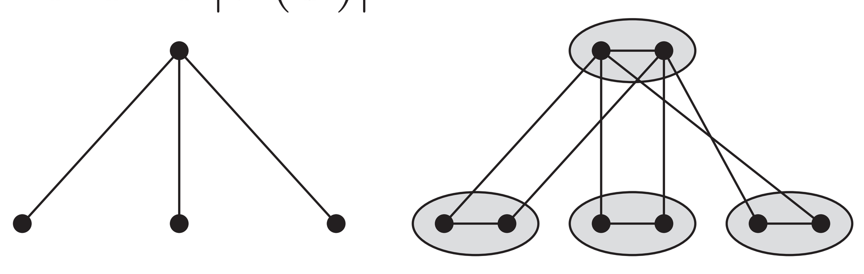
Covers

A *cover* of a graph G is a pair $\mathcal{H} = (L, H)$, where H is a graph and L is function $L : V(G) \to \mathcal{P}(V(H))$ such that the following conditions hold:

- The sets $\{L(v) : v \in V(G)\}$ form a partition of $V(H)$ and $H[L(v)]$ is a clique for each $v \in V(G)$;
- If $E_H(L(u), L(v)) \neq \emptyset$, then $u = v$ or $uv \in E(G)$, and if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

DP-Coloring

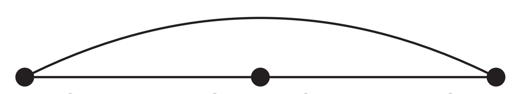
Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. A cover $\mathcal{H} = (L, H)$ of G is *k-fold* if $|L(v)| = k$ for each $v \in V(G)$. An *$\mathcal{H}$-coloring* is an independent set in H of size $|V(G)|$.



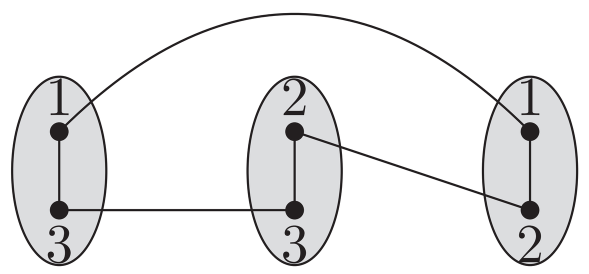
$K_{1,3}$ a 2-fold cover of $K_{1,3}$

- The *DP-chromatic number* $\chi_{DP}(G)$ is the smallest k such that G admits an $\mathcal{H}$-coloring for every k-fold cover $\mathcal{H}$ of G.
- Let $P_{DP}(G, \mathcal{H})$ be the number of $\mathcal{H}$- colorings of G. Then the *DP-color function* $P_{DP}(G, m)$ is the minimum value of $P_{DP}(G, \mathcal{H})$ over all m-fold covers $\mathcal{H}$ of G. For example, $P_{DP}(K_{1,3}, 2) = 2$.

Given a graph G and a k-assignment L, we can construct a cover $\mathcal{H}$ such that G admits an L coloring if and only if G admits an $\mathcal{H}$-coloring.



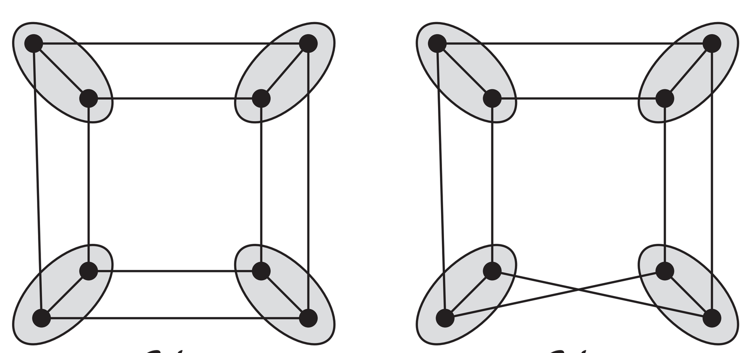
a 2-assignment for K_3



corresponding 2-fold cover of K_3

$$\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G) \leq \text{col}(G) \leq \Delta(G) + 1$$

Below are two distinct 2-fold covers of the 4-cycle C_4. Note that C_4 admits an $\mathcal{H}_1$-coloring but not an $\mathcal{H}_2$-coloring. In particular, $3 \leq \chi_{DP}(C_4) \leq \Delta(C_4) + 1 = 3$. This implies $\chi_{DP}(C_4) = 3$.

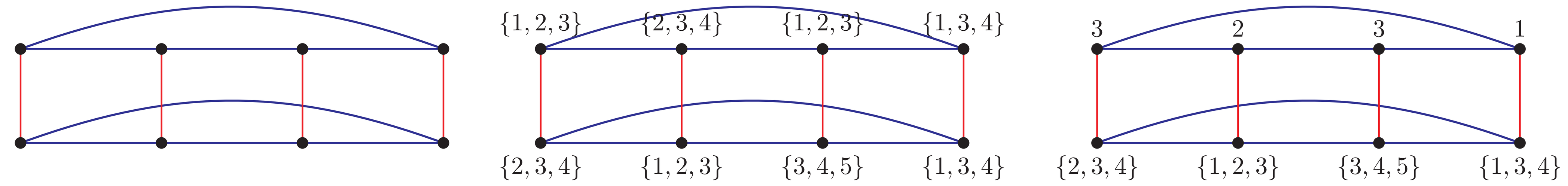


$\mathcal{H}_1$ $\mathcal{H}_2$

List Coloring the Cartesian Products of Graphs

The *Cartesian product* $G \square H$ of graphs G and H is a graph such that,

- $V(G \square H)$ is the Cartesian product of sets $V(G)$ and $V(H)$
- $(v_1, u_1)(v_2, u_2) \in E(G \square H) \iff$ either $v_1 = v_2$ and $u_1 u_2 \in E(H)$ or $u_1 = u_2$ and $v_1 v_2 \in E(G)$



$C_4 \square P_2$ $C_4 \square P_2$ $C_4 \square P_2$

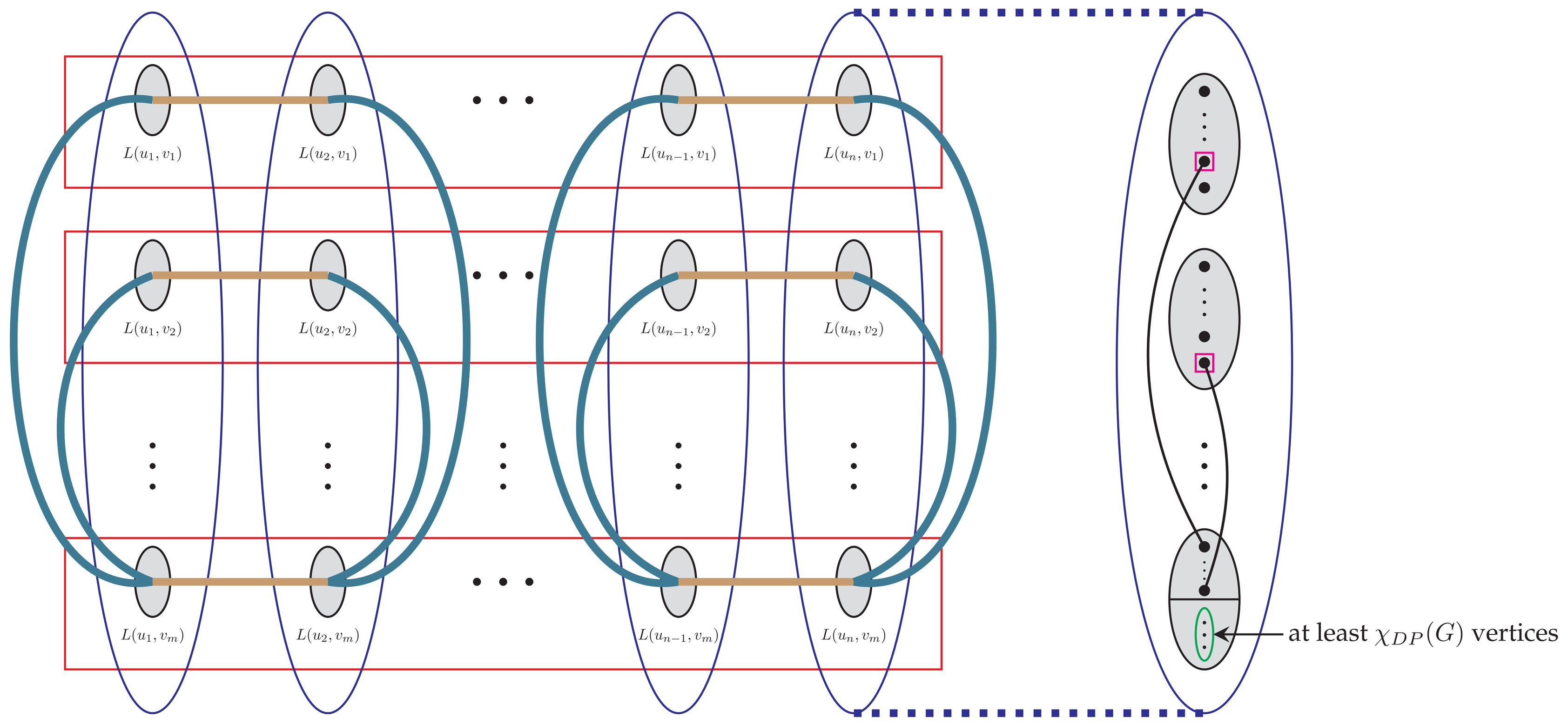
$\text{col}(P_2) = 2$
$\chi_\ell(C_4) = 2$

Theorem 1 (Borowiecki, et al. 2006). $\chi_\ell(G \square H) \leq \min\{\chi_\ell(G) + \text{col}(H), \chi_\ell(H) + \text{col}(G)\} - 1$.

DP-Coloring the Cartesian Products of Graphs

Our first result is to generalize the above theorem to the context of DP-coloring.

Theorem 2. $\chi_{DP}(G \square H) \leq \min\{\chi_{DP}(G) + \text{col}(H), \chi_{DP}(H) + \text{col}(G)\} - 1$.



Our next result shows that bound above is sharp when one of the factors is a complete bipartite graph.

Theorem 3. $\chi_{DP}(G \square K_{k,t}) = \chi_{DP}(G) + k$ *where* $t \geq (P_{DP}(G, k + \chi_{DP}(G) - 1))^k$.