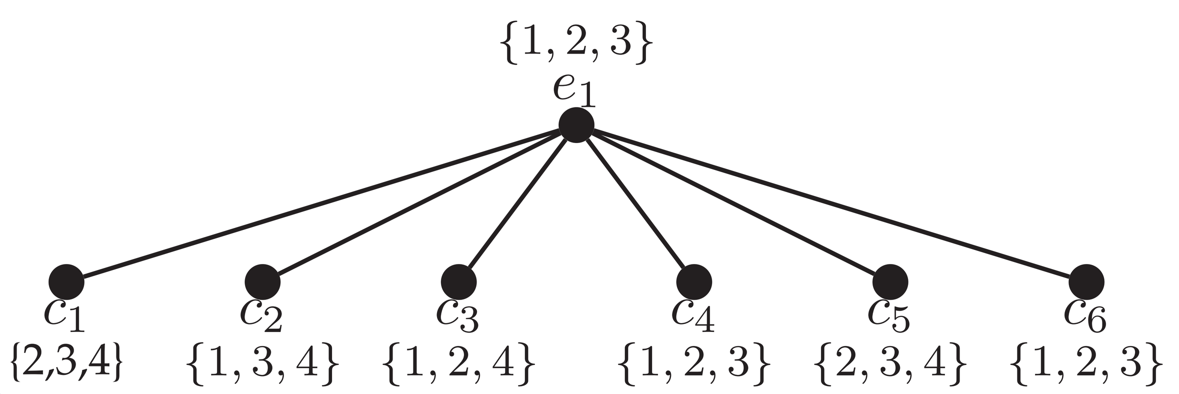
The Equitable Choosability of Complete Bipartite Graphs

Tim Wagstrom joint work with Jeffrey Mudrock, Madelynn Chase, Isaac Kadera, Ezekiel Thornburgh
College of Lake County Department of Mathematics; Grayslake, IL

1. Motivating Example

We need to schedule 1 engineering class and 6 calculus classes that meet every day for 50 minutes under the following constraints:

- The engineering class has to be at a different time than all of the calculus classes.
- 3 classrooms are available at each time.
- The professor for each class provides three times they are able to teach the class.



2. Equitable Coloring

A proper k-coloring, f, of a graph G is said to be an *equitable k-coloring* if the k color classes differ in size by at most 1. If f is an equitalbe k-coloring of graph G, it is easy to see that the size of each color class associated with f must be $\lceil |V(G)|/k \rceil$ or $\lfloor |V(G)|/k \rfloor$. We say that G is *equitably k-colorable* if there exists an equitable k-coloring of G.

Theorem 1 *Every graph G has an equitable k-coloring when $k \geq \Delta(G) + 1$.*

Conjecture 2 *A connected graph G is equitably $\Delta(G)$-colorable if it is different from K_m, C_{2m+1}, and $K_{2m+1,2m+1}$.*

Shown below is an equitable 4-coloring of P_6:

3. List Coloring

List coloring is a variation on the classic vertex coloring problem. Specifically, suppose we associate a list assignment, L, with a graph G such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$.

- Graph G is *L-colorable* if there is a proper coloring f of G where $f(v) \in L(v)$ for each $v \in V(G)$. We say f is a *proper L-coloring* of G.
- A list assignment L is a *k-assignment* for G if $|L(v)| = k$ for each $v \in V(G)$.
- Graph G is *k-choosable* if G is L-colorable whenever L is a k-assignment for G.

4. Equitable Choosability

In 2003 Kostochka et. al. introduced a list analog of equitable coloring. If L is a k-assignment for the graph G, a proper L-coloring of G is *equitable* if each color appears on at most $\lceil |V(G)|/k \rceil$ vertices. Graph G is *equitably L-colorable* if there is a proper L-coloring of G that is equitable. Graph G is *equitably k-choosable* if G is equitably L-colorable whenever L is a k-assignment for G.

Conjecture 3 *Every graph G is equitably k-choosable when $k \geq \Delta(G) + 1$.*

Conjecture 4 *A connected graph G is equitably k-choosable for each $k \geq \Delta(G)$ if it is different from K_m, C_{2m+1}, and $K_{2m+1,2m+1}$.*

5. Important Lemmas

Lemma 5 *Suppose that $G = \overline{K_m}$ and $L^{(1)}$ is a list assignment for G such that $|L^{(1)}(v)| \geq \eta$ for each $v \in V(G)$. If $\sigma \in \mathbb{N}$ is such that $m \leq \sigma\eta$, then there is a proper $L^{(1)}$-coloring of G that uses no color more than σ times.*

Lemma 6 *Suppose $G = K_{2,m}$ and the partite sets of G are $A' = \{u_1, u_2\}$ and $A = \{v_1, v_2, \ldots, v_m\}$. Also suppose that L is a k-assignment for G such that $L(u_1) \cap L(u_2) \neq \phi$. If $m \leq \lceil (m+2)/k \rceil (k-1)$ and $k < m+2$, then G is equitably L-colorable.*

Lemma 7 *Suppose that $G = K_{2,m}$ and the partite sets of G are $A' = \{u_1, u_2\}$ and $A = \{v_1, v_2, \ldots, v_m\}$. Also suppose that L is a k-assignment for G such that $L(u_1) \cap L(u_2) = \phi$. There must exist a $c_q \in L(u_1)$ and $c_r \in L(u_2)$ such that $|\{v \in A : \{c_q, c_r\} \subseteq L(v)\}| \leq m/4$.*

6. Results

Theorem 8 *$K_{n,m}$ is equitbly k-choosable if $m \leq \lceil (m+n)/k \rceil (k-n)$.*

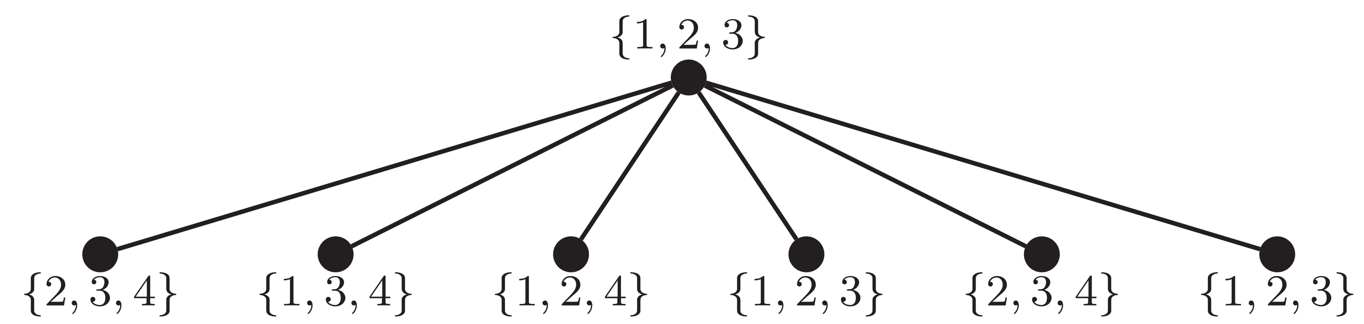
Theorem 9 *$K_{n,m}$ is not equitbly k-choosable if $m > \lceil (m+n)/k \rceil (k-1)$.*

Corollary 10 *$K_{1,m}$ is equitably k-choosable if and only if $m \leq \lceil (m+1)/k \rceil (k-1)$.*

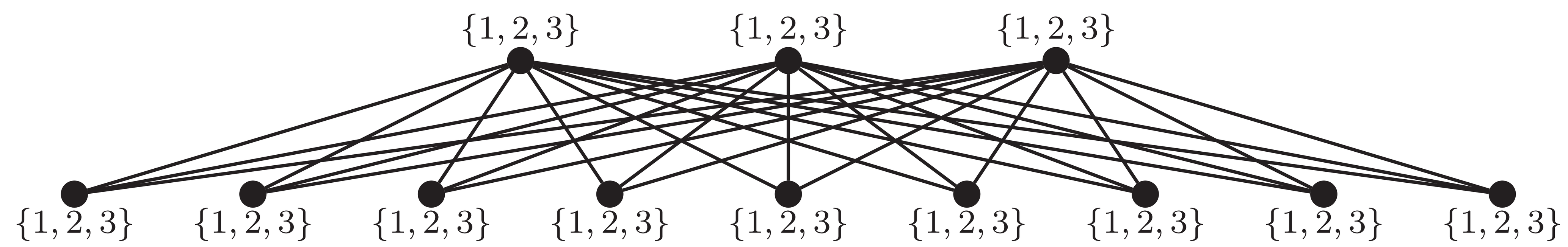
Theorem 11 *$K_{2,m}$ is equitably k-choossable if and only if $m \leq \lceil (m+2)/k \rceil (k-1)$.*

7. Proof Ideas and Examples

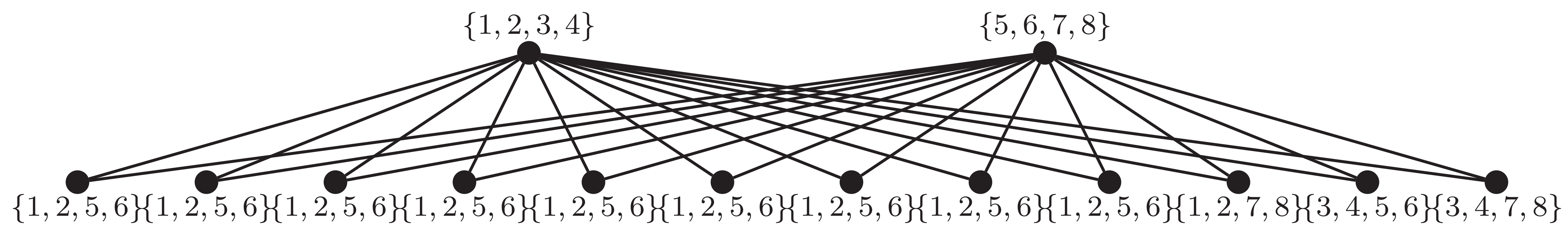
Example for Theorem 8: $n = 1$, $m = 6$, $k = 3$, $\lceil (m+n)/k \rceil = 3$.



Example for Theorem 9: $n = 3$, $m = 9$, $k = 3$, $\lceil (m+n)/k \rceil = 4$.



Example for Theorem 11: $n = 2$, $m = 12$, $k = 4$, $\lceil (m+n)/k \rceil = 4$.



Corollary 10 implies that $K_{1,25}$ is equitably k-choosable if and only if

$$k \in \{6, 8, 10, 11, 12\} \cup \{z \in \mathbb{N} : z \geq 14\}.$$

Also Theorem 11 implies that $K_{2,139}$ is equitably k-choosable if and only if

$$k \in \{14, 15, 17, 19, 20, 21, 22, 23\} \cup \{z \in \mathbb{N} : z \geq 25\}.$$

8. Future Research

- For $K_{n,m}$, study the smallest value $t \in \mathbb{N}$, at which $K_{n,m}$ is equitably k-choosable whenever $k \geq t$.
- We would like to characterize the equitable choosablity of $K_{n,m}$ for $n \geq 3$.
- We would like to study the equitable choosability of the disjoint union of complete bipartite graphs.