Research Statement

Jeffrey Mudrock
jmudrock@clcillinois.edu

1 Introduction

My research interests lie mostly in algebraic combinatorics, probabilistic combinatorics, and graph theory. Graph theory is the study of structures (called graphs) used to describe relations between objects. In a graph, the objects are called *vertices* and related objects are associated with (or connected by) an *edge*. Graphs can be used to model computer networks, social networks, biological networks, links between websites, or the bonding of atoms in a molecule. Unsurprisingly, the applications of graph theory are vast, and applications can be found in computer science, social science, biology, chemistry, physics, engineering, and economics (see [67] for graph theory basics and some applications). Broadly speaking, I am interested in applying techniques from algebraic combinatorics, probabilistic combinatorics, enumerative combinatorics, and extremal combinatorics to solve problems in graph theory. I am also interested in applications of my research to computer science (e.g., [40]).

Most of the research problems that I focus upon involve graph coloring. Classical vertex coloring models the ubiquitous problem of conflict-free allocation of resources. For example, imagine a graph where the vertices represent senate committees that meet every Friday, and we place an edge between any two committees that share at least one member. Then, we wish to "color" each vertex with a meeting time so that connected committees receive different meeting times. There are many well studied variations of classical vertex coloring. For example, in the senate committee example, imagine that we associate with each committee a list of times at which the committee is able to meet. Then, we wish to "color" each vertex with a meeting time from its corresponding list so that connected committees receive different meeting times. We will discuss below that this is called a list coloring problem, and we will also present a generalization of list coloring called DP-coloring.

As an undergraduate in the late 2000s, I studied graph labelings as a means of attacking the problem of decomposing classes of complete graphs, circulant graphs, and multigraphs, cyclically (see [9, 10, 13]). More recently, the results that I have obtained can be placed into three broad categories: list coloring and its equitable variants, DP-coloring, and enumerative list and DP-coloring. I have several research accomplishments of which I am quite proud: I gave a proof [51] that a *Discrete Mathematics* editor said might be from "The Book" [2], in [31] we have an argument featured in a book on the Combinatorial Nullstellensatz by Xuding Zhu and Rangaswami Balakrishnan [69], in [33] we discovered a connection between DP-coloring and a polynomial method for graph coloring, an undergraduate research project that I supervised [29] answered a 13 year old question of Carsten Thomassen [63], and in [32, 34, 39] we discovered a connection between a fundamental enumerative notion in list and DP-coloring and an extremal problem on coloring certain Cartesian products of graphs. Below, I provide an overview of some of my results and discuss my future research plans. Unless otherwise noted, all graphs mentioned in what follows are simple and finite, and we follow [67] for terminology and notation.

2 List Coloring and its Equitable Variants

In the classical vertex coloring problem, we wish to color the vertices of a graph G with up to m colors so that adjacent vertices receive different colors, a so-called *proper m-coloring*. When a proper m-coloring of G exists, it is called *m-colorable*. The *chromatic number* of a graph, denoted

$\chi(G)$, is the smallest m such that G is m-colorable. List coloring is a well known variation on classical vertex coloring that was introduced independently by Vizing [65] and Erdős, Rubin, and Taylor [15] in the 1970s. For list coloring, we associate a *list assignment* L with a graph G such that each vertex $v \in V(G)$ is assigned a list of available colors $L(v)$ (we say L is a list assignment for G). We say G is *L-colorable* if there is a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a *proper L-coloring* of G). A list assignment L for G is called a *k-assignment* if $|L(v)| = k$ for each $v \in V(G)$. A graph G is *k-choosable* if G is L-colorable whenever L is a k-assignment for G. The *list chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the smallest k such that G is k-choosable [1]. Since determining whether a graph is k-colorable is equivalent to determining whether it is L-colorable where L assigns the same k colors to every vertex of the graph, $\chi(G) \leq \chi_\ell(G)$ for every graph G.

2.1 Chromatic-Choosability and Strong Chromatic-Choosability

Graphs in which $\chi(G) = \chi_\ell(G)$ are known as *chromatic-choosable* graphs [58]. Determining whether a graph is chromatic-choosable is, in general, a challenging problem. Perhaps the most well known conjecture involving list coloring is about chromatic-choosability. Indeed, the famous Edge List Coloring Conjecture states that every line graph of a loopless multigraph is chromatic-choosable (see [26]). Many other conjectures and results about chromatic-choosability can be found in the literature (e.g., [23, 44, 57, 59, 64]).

In [51] I use Galvin's celebrated result that the list chromatic number of the line graph of any bipartite multigraph equals its chromatic number [23] to show the list packing number of any graph is well defined (see [11] for details on the list packing number of a graph). An editor of *Discrete Mathematics* remarked that he thinks my argument might be from "The Book" [2].

A graph G is called *k-critical* if $\chi(H) < \chi(G) = k$ whenever H is a proper subgraph of G. Criticality is a notion of fundamental importance in graph coloring since many coloring problems can be reduced to problems about critical graphs (see Section 5.2 in [67] for an overview). A couple of list analogues of criticality have appeared in the literature (see [12, 62]). In [34] we introduced the notion of strong k-chromatic-choosability. Strongly chromatic-choosable graphs include odd cycles, cliques, the join of a clique with any strongly chromatic-choosable graph, and many more infinite families of graphs. We use this notion to prove new results on chromatic-choosability and improve upon the best known upper bound on the list chromatic number of certain infinite families of graphs (see e.g., [32, 34]).

2.2 The Graph Polynomial and The Alon-Tarsi Number

One popular technique to solve problems in graph coloring is to encode combinatorial data about the graph of interest into a polynomial and then use algebraic techniques to prove facts about the polynomial (and hence the graph). For example, if G is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the *graph polynomial* of G is $f_G(x_1, \ldots, x_n) = \prod_{v_i v_j \in E(G), i<j}(x_i - x_j)$. One will note that if $f_G(x_1, \ldots, x_n) \neq 0$, the input corresponds to a proper coloring of G.

Alon and Tarsi used an algebraic theorem known as the Combinatorial Nullstellensatz to obtain a remarkable relationship between a special orientation of a graph and the graph polynomial. The result is the celebrated Alon-Tarsi Theorem, and it can be a valuable tool in bounding the list chromatic number of a graph (see [3, 4, 21]). Specifically, one of the implications of the Theorem is: if D is an orientation of the graph G with max indegree m and the number of even and odd

[1] It is well known that determining the chromatic number of a graph is NP-hard (see [24]). The decision problem of whether a graph is k-choosable is Π_2^p-complete if $k \geq 3$ (see [25]).

circulations contained in D differ, then $\chi_\ell(G) \leq m+1$ (note: a *circulation* of a digraph is a spanning subgraph such that the indegree and outdegree is the same at every vertex). In their seminal book on coloring problems Jensen and Toft [28] introduced a graph invariant related to the Alon-Tarsi Theorem. The *Alon-Tarsi number* of G, $AT(G)$, is the minimum m such that G has an orientation D with max indegree $m-1$ and the number of even and odd circulations contained in D differ

Zhu [68] and Schauz [61] introduced an online version of list coloring, and in the process, the graph invariant known as the *paint number*, denoted $\chi_p(G)$, was introduced. In 2010, Schauz [60] showed $\chi_p(G) \leq AT(G)$ for any graph G. In general, $\chi(G) \leq \chi_\ell(G) \leq \chi_p(G) \leq AT(G)$. So, whenever we use the Alon-Tarsi Theorem to bound the list chromatic number of a graph, we get the same bound on the paint number of the graph which is a stronger result. In [31] we use the Alon-Tarsi Theorem to show that the Cartesian product of an odd cycle or complete graph on at least 3 vertices and path is chromatic-choosable (meaning the three aforementioned inequalities become equality in this situation). Our argument is featured in a book on the Combinatorial Nullstellensatz by Zhu and Balakrishnan [69]. We also prove generalizations of this result to improve upon the best known bounds on the list chromatic number of other related Cartesian products.

2.3 Equitable Choosability

The study of equitable coloring began with a conjecture of Erdös [14] in 1964, but the general concept was formally introduced by Meyer [47] in 1973. An equitable k-coloring of a graph G is a proper k-coloring of G, f, such that the sizes of the color classes associated with f differ by at most one. A graph is said to be equitably k-colorable when an equitable k-coloring of G exists. In 2003, Kostochka, Pelsmajer, and West introduced a list version of equitable coloring [42] called *equitable choosability*. Suppose L is a k-assignment for the graph G. A proper L-coloring of G is *equitable* if each color appears on at most $\lceil |V(G)|/k \rceil$ vertices. Such a coloring is called an *equitable L-coloring* of G. A graph is *equitably k-choosable* if an equitable L-coloring of G exists whenever L is a k-assignment for G.

In 2018 and 2020, I supervised two undergraduate research projects that explored the equitable choosability of complete bipartite graphs. As a result of these projects, we were able to give a complete characterization of the equitable choosability of complete bipartite graphs that have a partite set of size at most 2 [52]. We also showed that determining whether an equitable L-coloring of the disjoint union of n stars exists when L assigns the same list of two colors to every vertex is NP-complete, and we showed there are infinitely many equitably 2-colorable graphs that are the disjoint union of two stars, but there are only 14 equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of two stars [40].

The *total graph* of graph G, denoted $T(G)$, is the graph with vertex set $V(G) \cup E(G)$ and vertices are adjacent if and only if the corresponding elements are adjacent or incident in G. In 1994, Fu [22] conjectured that for any graph G, $T(G)$ is equitably k-colorable whenever $k \geq \max\{\chi(T(G)), \Delta(G)+2\}$. In [36] we conjectured that something similar holds for equitable choosability: For every graph G, $T(G)$ is equitably k-choosable for each $k \geq \max\{\chi_\ell(T(G)), \Delta(G)+2\}$. We have proven that this Conjecture is true for all graphs of maximum degree at most 2, trees of maximum degree 3, stars, double stars, subdivisions of stars, and generalized theta graphs. To do this, we generalize a popular technique developed by Kostochka, Pelsmajer, and West [42] for proving equitable choosability. We also use our more general technique to prove that if G is a power of a cycle, then G is equitably k-choosable whenever $k \geq \Delta(G)$ (see [36, 54]). This proves that a well known conjecture of Kostochka, Pelsmajer, and West [42] holds for cycle powers.

In [37] we develop a new list analogue of equitable coloring called proportional choosability. Proportional choosability not only has have several nice properties that do not hold in the context of equitable choosability or equitable colorability (see [37, 38]), but it has also lead to some nice undergraduate research problems (see [53, 55]).

3 DP-Coloring

In 2015, Dvořák and Postle [20] introduced a generalization of list coloring called DP-coloring (they called it correspondence coloring) in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. DP-coloring has been extensively studied since its introduction (e.g., [5, 33, 46, 48]). Intuitively, DP-coloring is a generalization of list coloring where each vertex in the graph still gets a list of colors, but identification of which colors are different can change from edge to edge. For a graph G, the key object of study in DP-coloring is an m-fold cover of G which is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \to \{U : U \subseteq V(H)\}$ that satisfies particular conditions (see [6] for the precise definition). An *$\mathcal{H}$-coloring* of G is an independent set in H of size $|V(G)|$. The *DP-chromatic number* of G, $\chi_{DP}(G)$, is the smallest $m \in \mathbb{N}$ such that G has an $\mathcal{H}$-coloring whenever $\mathcal{H}$ is an m-fold cover of G.

Importantly, every m-assignment of graph G corresponds to an m-fold cover of G. However, for graphs G that contain at least one cycle, there are m-fold covers of G that don't correspond to any m-assignment of G. So, for any graph G we have that $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.

3.1 Probabilistic Techniques for the DP-Chromatic Number

If $k, t \in \mathbb{N}$ it is known that $\chi_{DP}(K_{k,t}) \leq k+1$. For each $k \in \mathbb{N}$, let $\mu(k)$ be the smallest natural number l such that $\chi_{DP}(K_{k,l}) = k+1$. In [49], I show that $\lceil k^k/k! \rceil \leq \mu(k) \leq 1+(k^k(\log(k!)+1))/k!$ which are currently the best known upper and lower bounds on $\mu(k)$. The proof of the upper bound is probabilistic and relies on an alteration of a randomly constructed cover.

Suppose $G \square H$ is the Cartesian product of graphs G and H. In [39], we show that $\chi_{DP}(G \square H) \leq \min\{\chi_{DP}(G) + \text{col}(H), \chi_{DP}(H) + \text{col}(G)\} - 1$ where $\text{col}(H)$ is the coloring number of the graph H. We also use partially random processes for lower bound arguments for $\chi_{DP}(G \square K_{k,t})$ and use them to show the sharpness of this bound.

3.2 An Algebraic Method for DP-Coloring

Since there are striking similarities and differences between DP-coloring and list coloring, it is natural to ask whether the Combinatorial Nullstellensatz or an analogue of the Alon-Tarsi Theorem (see Subsection 2.2) can be applied to DP-coloring. As many researchers have noticed, for any $n \in \mathbb{N}$ the counterclockwise orientation of the edges of a copy of C_{2n+2}, D, has 2 even circulations and 0 odd circulations. Moreover, the indegree of every vertex in D is 1 which means $\chi_\ell(C_{2n+2}) \leq 2$ by the Alon-Tarsi Theorem. However, $\chi_{DP}(C_{2n+2}) = 3$. So, the Alon-Tarsi Theorem doesn't hold in general for the DP-chromatic number.

In [33] we discovered a way to apply the Combinatorial Nullstellensatz to DP-coloring. The key is to view graph polynomials as polynomials over some appropriate finite field rather than $\mathbb{R}$. We apply our ideas to DP-coloring of the cones of certain bipartite graphs and uniquely 3-colorable graphs. We also extend a result of Akbari, Mirrokni, and Sadjad [1] on unique list colorability to the context of DP-coloring, and we establish a sufficient algebraic condition for a graph G to satisfy $\chi_{DP}(G) \leq 3$.

4 Enumerative List and DP-coloring

In 1912, Birkhoff [7] introduced the notion of the chromatic polynomial in hopes of using it to make progress on the four color problem. For $m \in \mathbb{N}$, the *chromatic polynomial* of a graph G, denoted $P(G,m)$, is the number of proper m-colorings of G. It is easy to show that $P(G,m)$ is a polynomial in m of degree $|V(G)|$.

4.1 The List Color Function

The notion of chromatic polynomial was extended to list coloring in the early 1990s by Kostochka and Sidorenko [43]. If L is a list assignment for graph G, let $P(G,L)$ denote the number of proper L-colorings of G. The *list color function* $P_\ell(G,m)$ is the minimum value of $P(G,L)$ where the minimum is taken over all possible m-assignments L for G. Since an m-assignment could assign the same m colors to every vertex in a graph, it is clear that $P_\ell(G,m) \leq P(G,m)$ for each $m \in \mathbb{N}$. In general, the list color function can differ significantly from the chromatic polynomial for small values of m. On the other hand, in 1992, Donner [16] showed that for any graph G there is a $k \in \mathbb{N}$ such that $P_\ell(G,m) = P(G,m)$ whenever $m \geq k$.

With Donner's 1992 result in mind, it is natural to study the point at which the list color function of a graph becomes identical to its chromatic polynomial. Given any graph G, the *list color function number of* G, denoted $\nu(G)$, is the smallest $t \geq \chi(G)$ such that $P_\ell(G,t) = P(G,t)$. The *list color function threshold of* G, denoted $\tau(G)$, is the smallest $k \geq \chi(G)$ such that $P_\ell(G,m) = P(G,m)$ whenever $m \geq k$. Clearly, $\chi(G) \leq \chi_\ell(G) \leq \nu(G) \leq \tau(G)$.

One of the most famous and important open questions on the list color function relates to whether the list color function number of a graph can differ from its list color function threshold.

Question 1 (Kirov and Naimi [41]) *If $P_\ell(G,t) = P(G,t)$ for some $t \geq \chi(G)$, does it follow that $P_\ell(G,t+1) = P(G,t+1)$?*

Much of the research on the list color function has been focused on studying the list color function threshold. In 2009, Thomassen [63] showed that for any graph G, $\tau(G) \leq |V(G)|^{10} + 1$. Then, in 2017, Wang, Qian, and Yan [66] showed that for any graph G, $\tau(G) \leq (|E(G)|-1)/\ln(1+\sqrt{2}) + 1$. Then, in 2022, Dong and Zhang [19] showed $\tau(G) \leq |E(G)| - 1$ when $|E(G)| \geq 4$. Thomassen also asked the following question in his 2009 paper.

Question 2 (Thomassen [63]) *Is there a universal constant α such that for any graph G, $\tau(G) - \chi_\ell(G) \leq \alpha$?*

Question 2 went unanswered for 13 years until the following was shown in an undergraduate research project that I supervised [29]: there is a positive constant C such that $\tau(K_{2,n}) - \chi_\ell(K_{2,n}) \geq C\sqrt{n}$ whenever $n \geq 16$. Notice that this shows that the answer to Question 2 is no in a rather strong sense. I also supervised an undergraduate research project that focused on bounding $\tau(K_{2,n})$ from above [30].

4.2 The DP Color Function

Following Kostochka and Sidorenko's definition of the list color function, Kaul and I introduced a DP-coloring analogue of the chromatic polynomial [35] which has attracted interest from other researchers (e.g., [17, 18, 45]). Suppose $\mathcal{H} = (L,H)$ is a cover of graph G. Let $P_{DP}(G,\mathcal{H})$ be the number of $\mathcal{H}$-colorings of G. Then, the *DP color function* of G, denoted $P_{DP}(G,m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible m-fold covers $\mathcal{H}$ of G. It is easy to show that for any graph G and $m \in \mathbb{N}$, $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

The list color function and DP color function of certain graphs behave similarly. However, for some graphs there are surprising differences. For example, I supervised an undergraduate research project where we showed that the DP color function analogue of Question 1 has an answer of no [8]. Similar to the list color function, $P_{DP}(G, m) = P(G, m)$ for every $m \in \mathbb{N}$ whenever G is chordal or an odd cycle. On the other hand, if G is a graph with girth that is an even number, then there exists an $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$ (see [35]). This result is particularly interesting since we know that the list color function of any graph eventually equals its chromatic polynomial. As part of an undergraduate research project, we were able to determine the largest possible asymptotic behavior of $P(G, m) - P_{DP}(G, m)$ as $m \to \infty$.

Theorem 3 ([56]) *Suppose g is an odd integer with $g \geq 3$. If G is a graph on n vertices with girth g or $g+1$, then $P(G, m) - P_{DP}(G, m) = O(m^{n-g})$ as $m \to \infty$. Consequently, $P(M, m) - P_{DP}(M, m) = O(m^{|V(M)|-3})$ as $m \to \infty$ for any graph M.*

We were also able to demonstrate the tightness of Theorem 3 for all possible girths, and we showed that any graph with a dominating vertex must have a DP color function that eventually equals its chromatic polynomial.

Since the DP color function of a graph need not eventually equal its chromatic polynomial, the following is an important algebraic open question on the DP color function.

Question 4 ([27]) *Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$?*

As part of an undergraduate research project, we showed that the answer to Question 4 is yes if we restrict our attention to graphs that have a vertex whose removal yields an acyclic graph [27]. In order to make potential progress on questions like Question 4, I also introduced a deletion-contraction relation for the DP color function which requires extending the definition of the DP color function to multigraphs and the introduction of the dual DP color function of a graph which counts the maximum number of DP-colorings of a graph over certain covers [50].

An interesting application of the DP color function is its utility for the following extremal question: What is the smallest t for which $\chi_{DP}(G \square K_{k,t}) = \chi_{DP}(G) + k$ (see Subsection 3.1 for this question's motivation)? In [39], we illustrate that the DP color function of G is essential in the study of this question. Similarly, the list color function is essential to the study of the analogous question in the list coloring context (see [32, 34]).

5 Future Work

My research experience makes me well suited to pursue a wide variety of problems in algebraic combinatorics, probabilistic combinatorics, and graph theory. I am currently working on several exciting projects. Anton Bernshteyn, Hemanshu Kaul, and I are working with a doctoral student on extending the notion of strong chromatic choosability to the context of DP-coloring, and we are working with another doctoral student on DP-coloring of graphs with random covers. Also, Hemanshu Kaul, Samantha Dahlberg, and I are working on using the algebraic ideas described in Subsection 3.2 to find new bounds on the DP color function of graphs.

All in all, I feel well prepared to sustain continued scholarship. I am confident I will keep developing research questions accessible to undergraduate and graduate students, and I am excited to continue to work with students.

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