Proportional 2-Choosability of Graphs With a Restricted Palette

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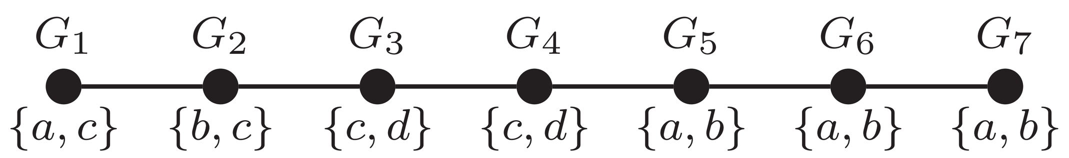
1. Motivating Example

For an eight-team single-elimination-style basketball tournament, we must assign each of the seven games one of four available referee crews. It turns out that two of the four crews are available per game. We must assign a crew to each game under the following restrictions:

- No crew referees two consecutive games.
- Any crew available for $2k$ games must referee k games.

Shown below is the tournament as represented by a path on seven vertices (denoted by P_7).



2. Equitable Coloring

Suppose we assign a color to each vertex of a graph G on n vertices.

- We have a **proper coloring** of G if whenever two vertices are connected by an edge, the vertices receive different colors.
- A **proper k-coloring** of G is a proper coloring of G that uses no more than k colors.
- An **equitable k-coloring** of G is a proper k-coloring where each of the k colors is used $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ times.

Shown below is an equitable 4-coloring of P_6:

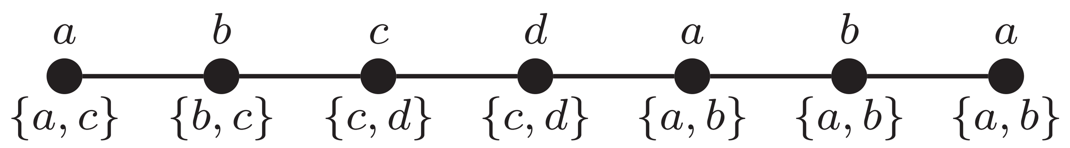
3. List Coloring

- A **list assignment** L of a graph G is a function that associates with each vertex of G a list of colors.
- L is a **k-assignment** for a graph G if L associates with each vertex of G a list of colors of size k.
- A **proper L-coloring** of G is a proper coloring of G where each vertex v is assigned a color in $L(v)$.
- If, for each k-assignment L for G, we can find a proper L-coloring of G, we say G is **k-choosable**.

4. Equitable List Coloring

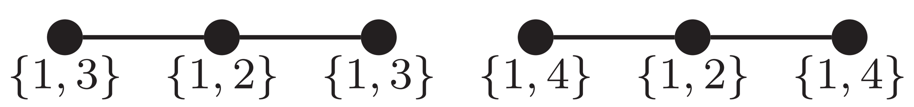
In 2003, Kostochka, Pelsmajer, and West introduced a list analogue of equitable coloring called equitable choosability. Throughout this section, suppose L is a k-assignment for a graph G on n vertices. An **equitable L-coloring** of G is a proper L-coloring of G where each color appears on at most $\lceil n/k \rceil$ vertices.

Shown below is an equitable L-coloring of P_7, where L is the same list assignment from our motivating example.



In 2019, Kaul, Mudrock, Pelsmajer, and Reiniger introduced a new type of equitable list coloring called proportional choosability. The **multiplicity** of a color c, denoted by $\eta(c)$, is the number of lists in which the color occurs. The **palette** of L, denoted by $\mathcal{L}$, is the union of all the lists in the range of L. A **proportional L-coloring** of G is a proper L-coloring of G where each color $c \in \mathcal{L}$ appears on either $\lceil \eta(c)/k \rceil$ vertices or $\lfloor \eta(c)/k \rfloor$ vertices. G is **proportionally k-choosable** if we can find a proportional L-coloring of G whenever L is a k-assignment for G.

2-assignment for a graph consisting of two disconnected paths, each on 3 vertices (denoted by $P_3 + P_3$):



$\eta(1) = 6$ - use three times
$\eta(i) = 2$ for $i = 2, 3, 4$ - use once

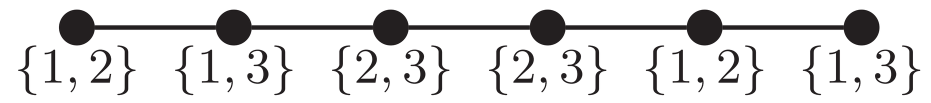
5. Restricted Palette

Theorem 1 *G is proportionally 2-choosable if and only if G is a graph consisting of disconnected paths such that the largest component of G has at most 5 vertices and all other components of G have 2 or fewer vertices.*

- A list assignment L is a **(k,ℓ)-assignment** for G if L is a k-assignment for G where the palette of L has size no more than ℓ.
- G is **proportionally (k,ℓ)-choosable** if we can find a proportional L-coloring of G whenever L is a (k,ℓ)-assignment for G.

Theorem 2 *G is proportionally (k,k)-choosable if and only if G is equitably k-colorable.*

Theorem 3 *If G is proportionally $(k,\ell+1)$-choosable, then G is proportionally (k,ℓ)-choosable.*

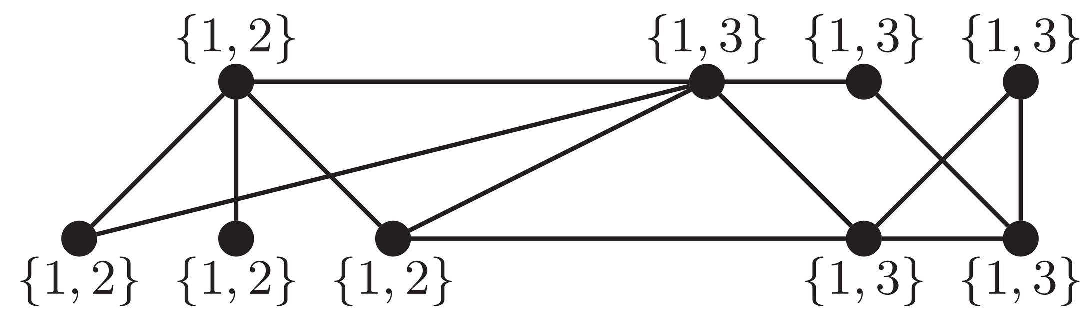


6. Results

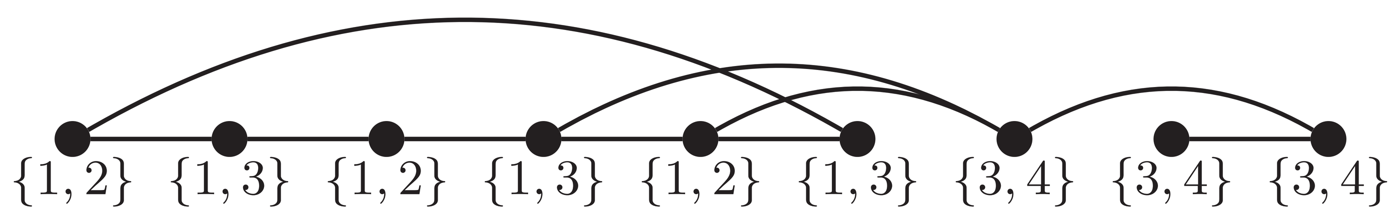
Our goal was to characterize the proportionally $(2,\ell)$-choosable graphs for each $\ell \geq 2$.

Corollary 4 *G is proportionally $(2,2)$-choosable if and only if G is equitably 2-colorable.*

Proposition 5 *If there exists a vertex in G connected to more than two other vertices in G, then G is not proportionally $(2,\ell)$-choosable for $\ell \geq 3$.*

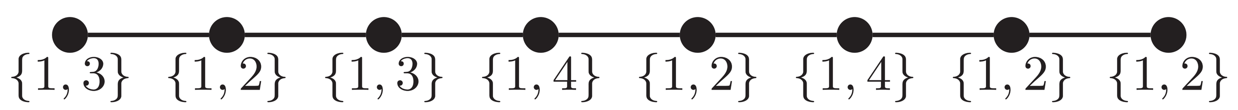


Proposition 6 *No cycle is proportionally $(2,3)$-choosable. Furthermore, if a graph contains a cycle, then it is not proportionally $(2,\ell)$-choosable for $\ell \geq 4$.*



Proposition 7 *If a graph contains $P_3 + P_3$, then it is not proportionally $(2,\ell)$-choosable for $\ell \geq 5$.*

Proposition 8 *P_n is not proportionally $(2,4)$-choosable for $n = 6$ and each $n \geq 8$.*



Theorem 9 *A connected graph G is proportionally $(2,3)$-choosable if and only if $G = P_n$.*

Theorem 10 *A connected graph G is proportionally $(2,4)$-choosable if and only if $G = P_7$ or $G = P_n$ for $n \leq 5$.*

Theorem 11 *G is proportionally $(2,\ell)$-choosable for $\ell \geq 5$ if and only if G is proportionally 2-choosable.*