Equitable Choosability of the Disjoint Union of Stars

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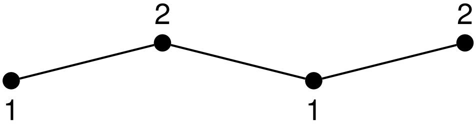
11/6/2020

Joint work with Hemanshu Kaul and Jeffrey Mudrock

Classical Coloring

- A ***proper k-coloring*** of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$ and $f(u) \neq f(v)$ whenever u and v are adjacent in G.



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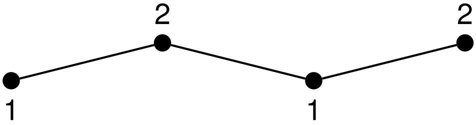

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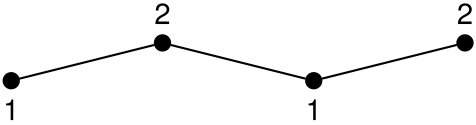

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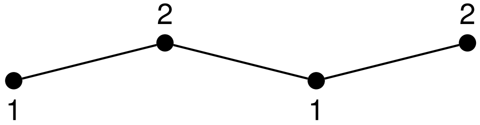

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- Note that the color classes are independent sets.

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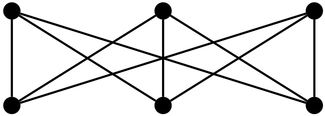

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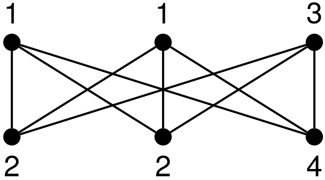

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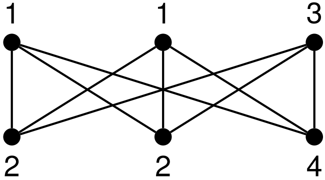

Figure: Is $G = K_{3,3}$ equitably 4-colorable? Yes. We have that $\lfloor 6/4 \rfloor = 1$ and $\lceil 6/4 \rceil = 2$.

- Note that $K_{3,3}$ is equitably 2-colorable but not equitably 3-colorable.

Important Theorems and Conjectures for Equitable Coloring

Theorem (Hajnal and Szemeredi(1970))

Every graph G has an equitable k-coloring when $k \geq \Delta(G) + 1$.

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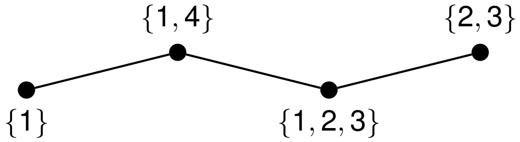

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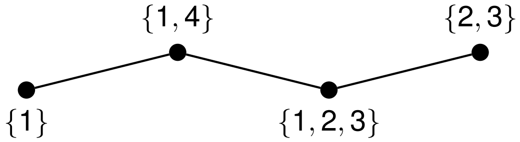

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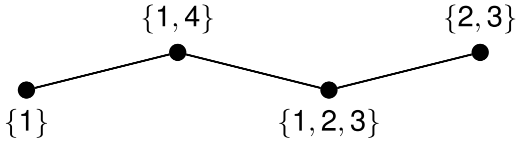

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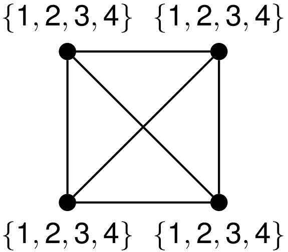
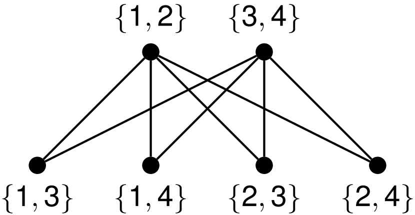
- For example, the complete graph K_n is n-choosable. Also, $K_{2,4}$ is not 2-choosable.





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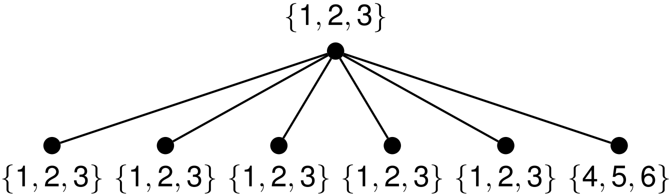

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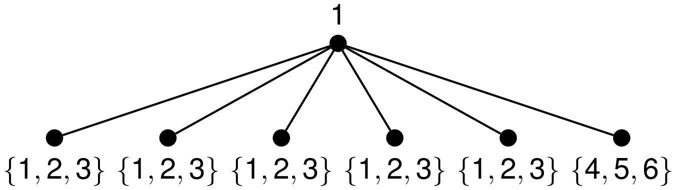

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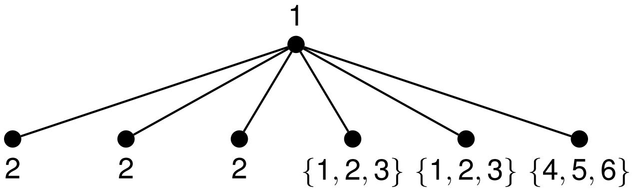

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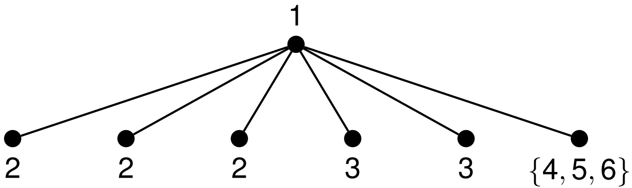

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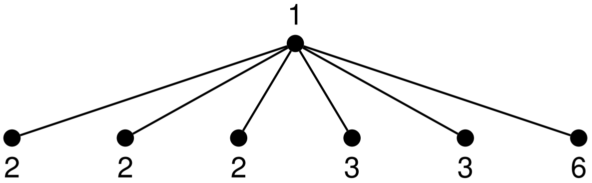

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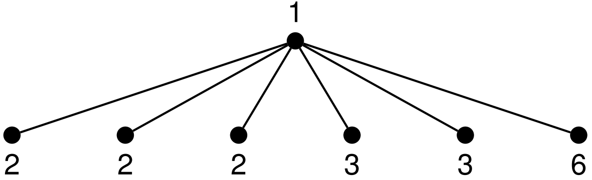

- Unlike equitable coloring, our only concern in equitable choosability is not overusing any color.

Equitable Choosability Conjectures

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Results for $k < \Delta(G)$

Theorem (Kostochka, Pelsmajer, and West (2003))

If G is a forest and $k \geq 1 + \Delta(G)/2$, then G is equitably k-choosable. Also for all D there is a tree with maximum degree at most D that is not equitably $\lceil D/2 \rceil$-choosable.

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Conjecture (Kaul, Mudrock, and Pelsmajer (2018))

Let $T(G)$ denote the total graph of G. For every graph G, $T(G)$ is equitably k-choosable for each $k \geq \max\{\chi_\ell(T(G)), \Delta(T(G))/2 + 2\}$.

Characterizations

Theorem (Mudrock, Chase, Kadera, Thornburgh, W. (2018))

$K_{1,m}$ is equitably k-choosable if and only if $m \leq \lceil (m+1)/k \rceil (k-1)$.

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Suppose that $G_1, G_2 \ldots G_n$ are pairwise vertex disjoint graphs and $G = \sum_{i=1}^{n} G_i$. If G_i has an equitable k-coloring for all $i = 1, 2, \ldots, n$ then G has an equitable k-coloring.

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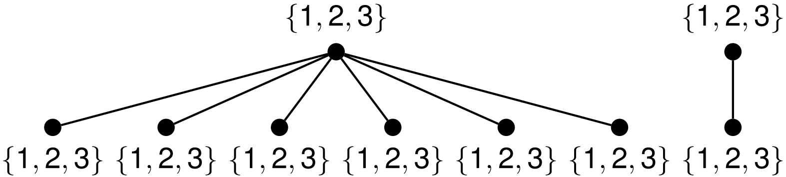

Figure: Both $K_{1,6}$ and $K_{1,1}$ are equitably 3-choosable. However, $K_{1,6} + K_{1,1}$ is not equitably 3-choosable

Motivating Question

Question

Suppose that $n \geq 2$. For which $k, m_1, m_2, \ldots, m_n \in \mathbb{N}$ is $\sum_{i=1}^{n} K_{1,m_i}$ equitably k-choosable?

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- If a graph is not equitably 2-colorable then it is also not equitably 2-choosable.

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Lemma

There is a partition $\{A, B\}$ of the set $[n]$ such that $\sum_{i \in A} m_i = \sum_{j \in B} m_j$ if and only if $G = \sum_{i=1}^{n} K_{1, m_i + 1}$ is equitably 2-colorable.

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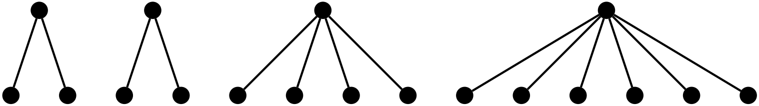

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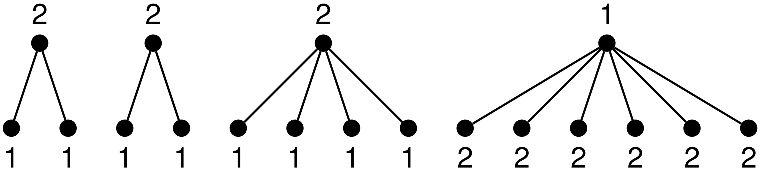

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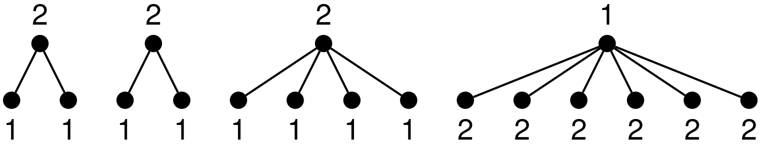

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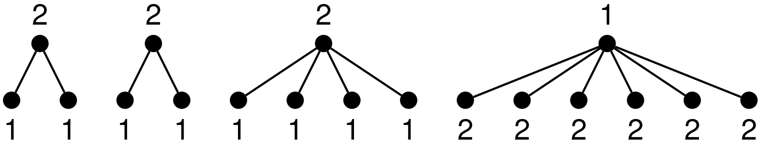

- We want to find a partion $\{A, B\}$ of [4] such that

$$\sum_{i \in A} m_i = \sum_{j \in B} m_j.$$

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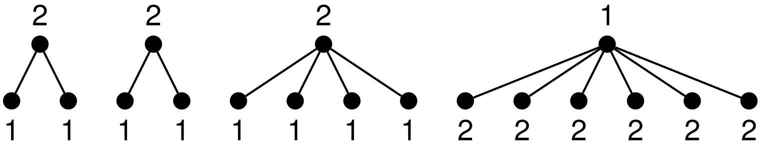

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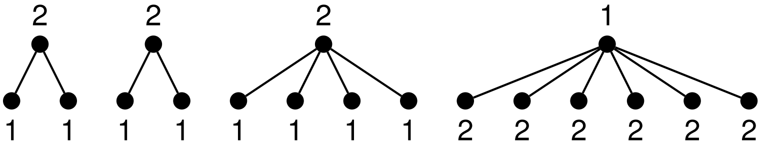

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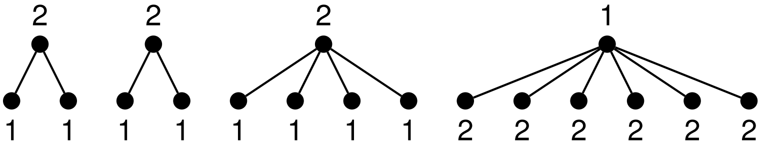

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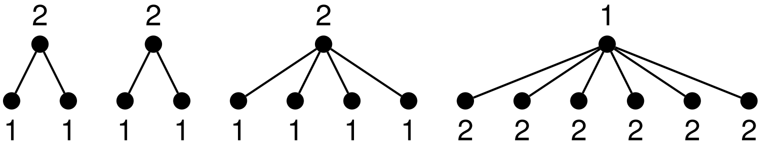

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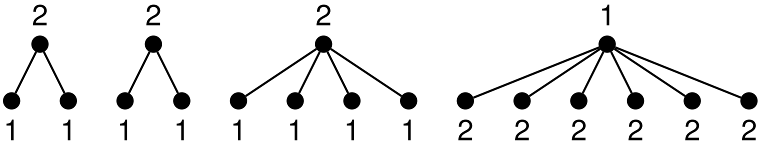

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Equitable 2-choosability

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Let $G = K_{1,m_1} + K_{1,m_2}$ where $1 \leq m_1 \leq m_2$. G is equitably 2-choosable if and only if $m_2 - m_1 \leq 1$ and $m_1 + m_2 \leq 15$.

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Suppose that $n \geq 2$. Are there only finitely many equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of n stars?

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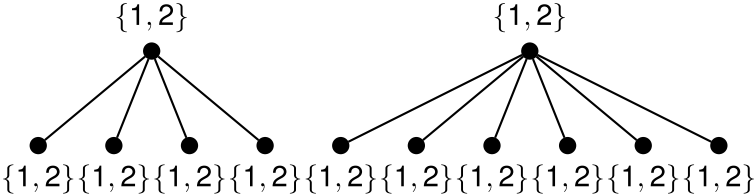
$m_2 - m_1 > 1$

- Consider the following example:



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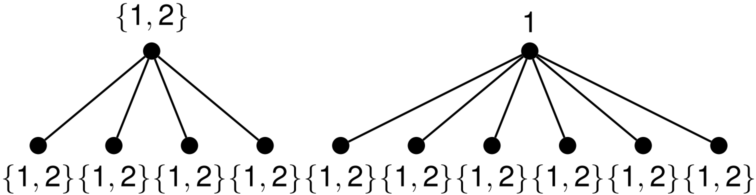

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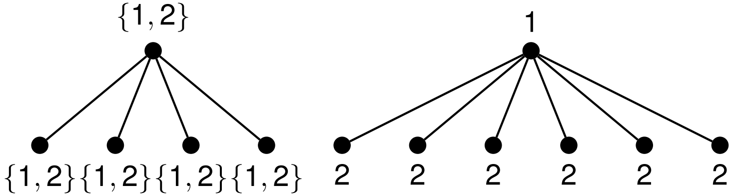

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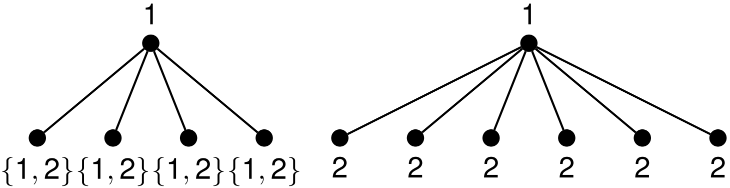

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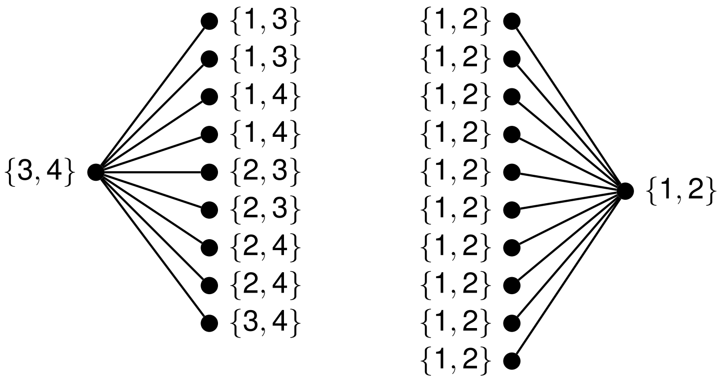

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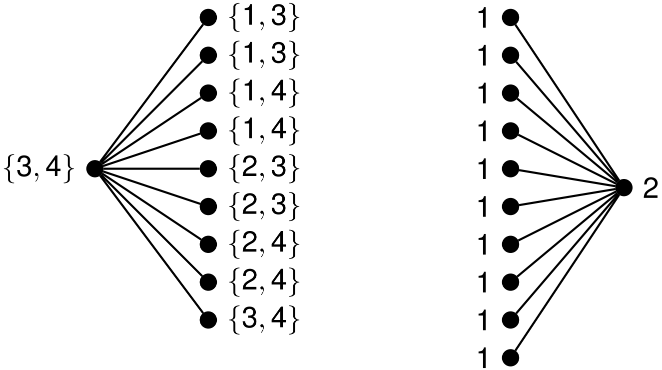

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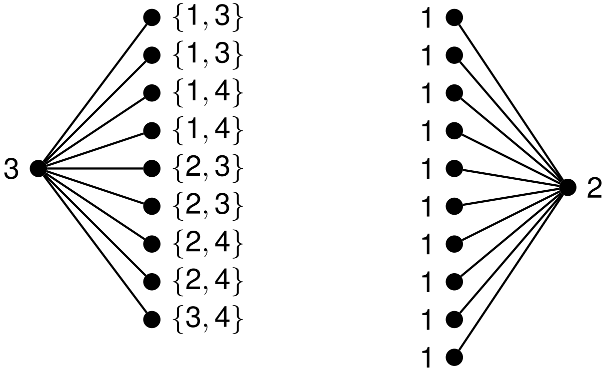

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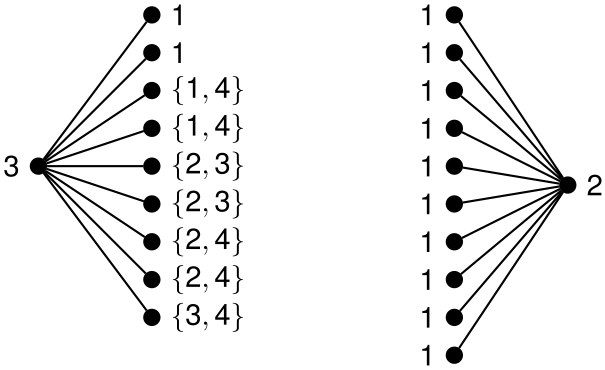

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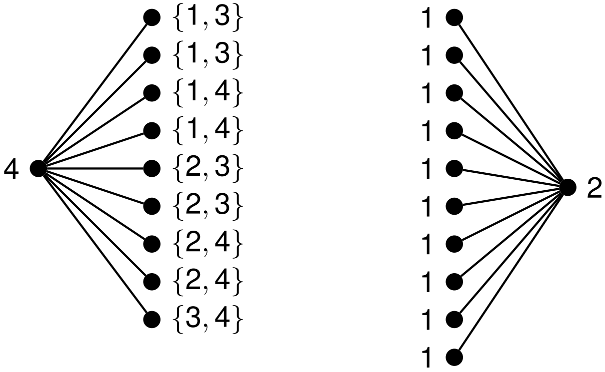

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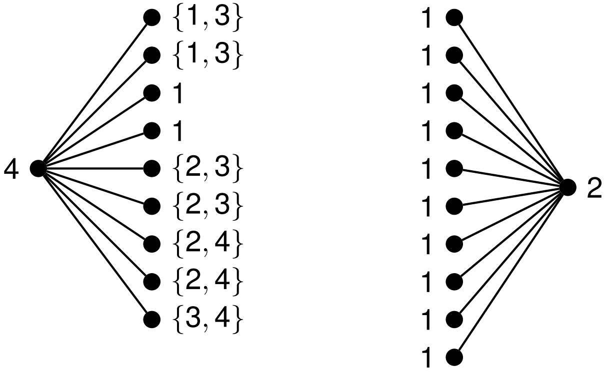

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Let $G = K_{1,m} + K_{1,m}$ where $m \in [7]$. Then G is equitably 2-choosable.

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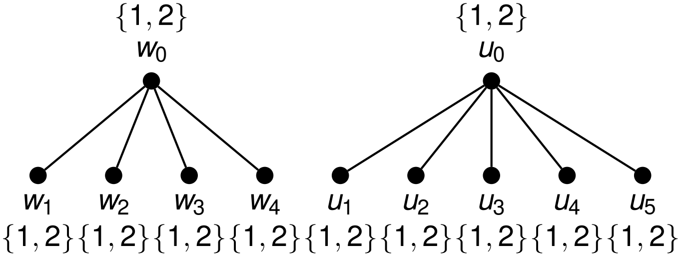

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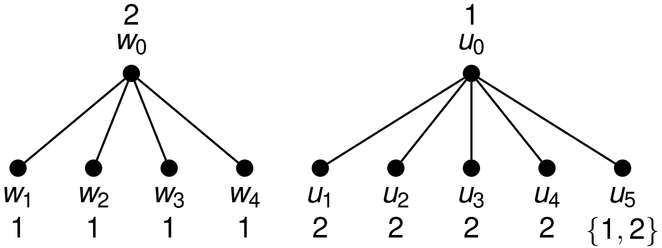

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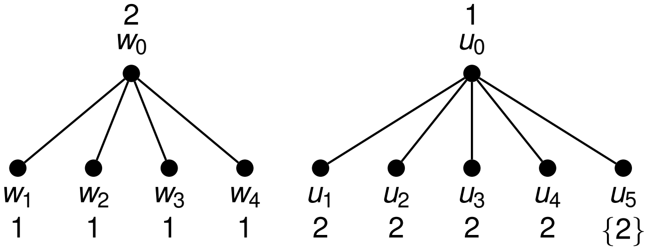

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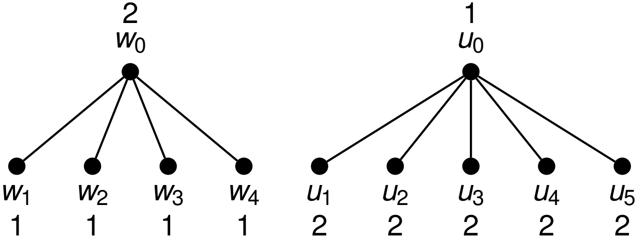

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Suppose that $n, m \in \mathbb{N}$, $n \geq 2$, and that $G = \sum_{i=1}^{n} K_{1,m}$. When n is odd, G is equitably 2-choosable if and only if $m \leq 2$. When n is even, G is equitably 2-choosable if and only if $m \leq 7$.

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Equitable k-Choosability

Theorem (Kaul, Mudrock, and W. (2020))

Let $k \in \mathbb{N}$ and $m_1 \leq m_2$.
If $m_2 \leq \lceil (m_2 + m_1 + 2)/k \rceil (k-1) - 1$
and $m_1 + m_2 \leq 15 + \lceil (m_2 + m_1 + 2)/k \rceil (k-2)$ then
$K_{1,m_1} + K_{1,m_2}$ is equitably k-choosable.

Necessity of the First Condition

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- recall the second condition

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$$\begin{aligned} 8 + 9(k-1) - 1 &\leq 15 + \lceil (8 + 9(k-1) - 1 + 2)/k \rceil (k - 2) \\ 9(k-1) + 7 &\leq 15 + 9(k-2) \\ 9k - 2 &\leq 9k - 3 \end{aligned}$$

Improving the Second Condition cont.

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$$\begin{aligned} k^4 - k^2 + 3k - 2 &\leq 15 + \left\lceil \frac{k^3 + (k-1)(k^3 - k + 2) + 2}{k} \right\rceil (k-2) \\ k^4 - k^2 + 3k - 2 &\leq 15 + (k^3 - k + 1)(k-2) \\ k^4 - k^2 + 3k - 2 &\leq k^4 - 2k^3 - k^2 + 3k + 13 \\ 2k^3 &\leq 15 \end{aligned}$$

Proof of Equitable k-Choosability

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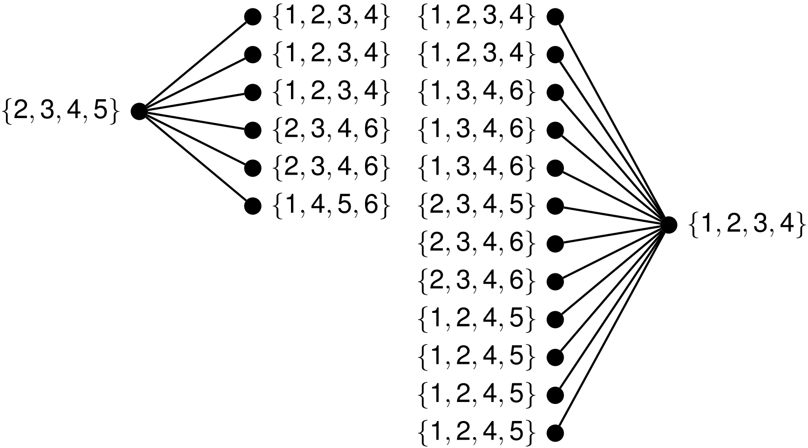
- We use this to justify the existence of the extremal choice for the partial list colorings.

Example of ϵ-greedy process

- We will demonstrate how the ϵ-greedy process works with the following example.



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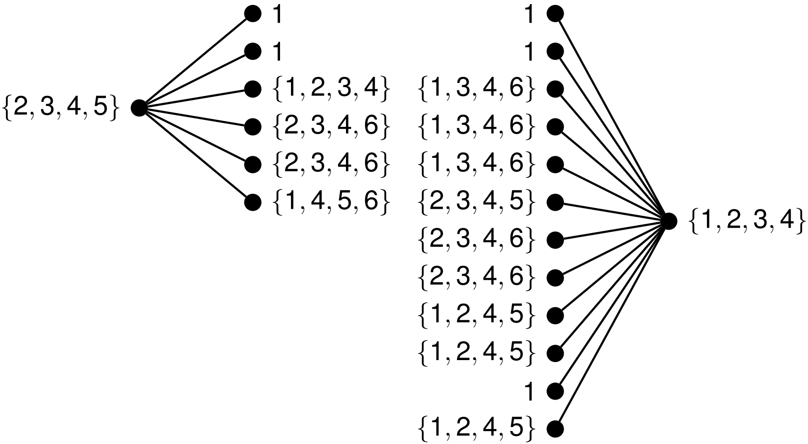

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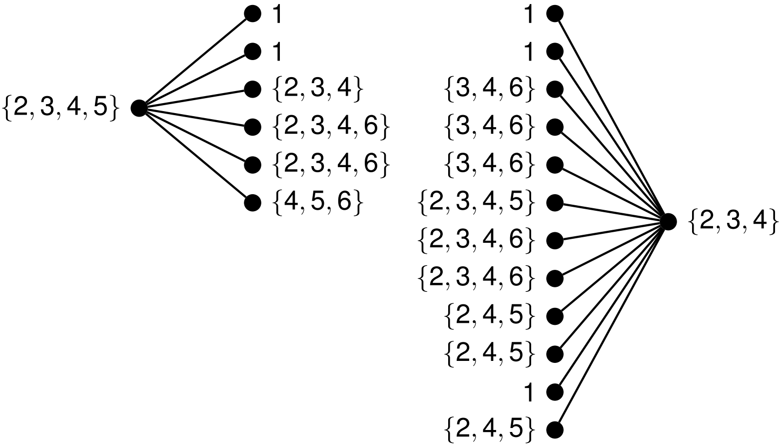

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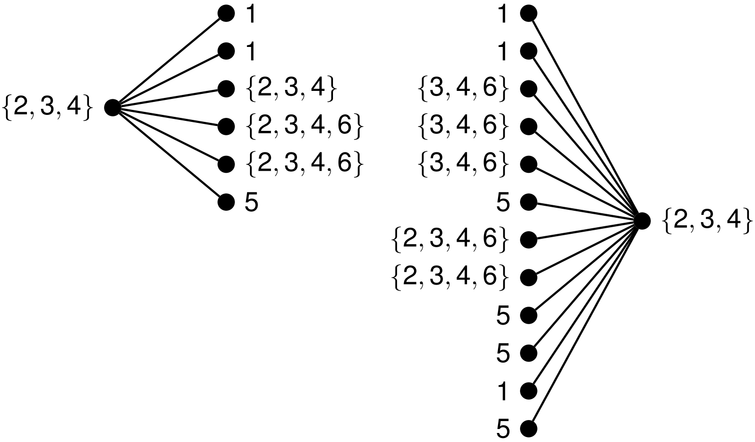

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- This extremal choice lets us apply the previous theorem to help finish this coloring.

Questions

- Questions?

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Suppose that $n \geq 2$. For which $k, m_1, m_2, \ldots, m_n \in \mathbb{N}$ is $\sum_{i=1}^{n} K_{1,m_i}$ equitably k-choosable?

Question

Is STARS EQUITABLE 2-CHOOSABLITY NP-hard?

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Suppose that $n \geq 2$. Are there only finitely many equitably 2-choosable graphs (up to isomorphism) that are the disjoint union of n stars?